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On the Thermal Property of Arbitrarily Accelerating Charged Black Hole with a New Tortoise Coordinate Transformation

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After a new tortoise coordinate transformation is adopted, the entropy and non-thermal radiation of an arbitrarily accelerating charged black hole are discussed as an example of non-stationary black holes. The same cut-off relation is chosen as static case, which is independent of space-time, and then the entropy of the non-stationary black hole is also proportional to the area of its event horizon. Meanwhile, the crossing of the particle energy levels near the event horizon is studied, the representative of the maximum value of the crossing energy levels is the same as the usual tortoise coordinate transformation.

KEY WORDS: entropy; black hole; radiation; brick wall model.

1. INTRODUCTION

Since Bekenstein suggested that the entropy of a black hole is proportional to its surface area, the related research work has got much progress (Bekenstein, 1973; Hawking, 1975). The brick-wall model suggested by ‘t Hooft (1985) has been studied extensively for its relation to the statistical explanation of black hole entropy. Recently, the brick-wall model is improved to the thin film model (Li and Zhao, 2001). Adopting the usual tortoise coordinate transformation, in process of calculating the static or stationary black hole’s entropy by the thin film model, the cut-off relation become more simpler. However, in the non-stationary black hole space-time, the cut-off relation is very complicated, and it varies with the space-time if we want to keep the Bekenstein–Hawking entropy is proportional to the area (He *et al.*, 2002). In this paper, using a new tortoise coordinate transformation (Yang, 2003), we calculate the entropy of an arbitrarily accelerating charged

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black hole as an example of non-stationary black holes, successfully simplify its complicated cut-off relation to a simple one which is the same as the cut-off relation in static case and preserve the conclusion that the entropy of black hole is proportional to its surface area. Meanwhile, using the new tortoise coordinate transformation, we discuss the non-thermal radiation of the black hole and find that the representative of the maximum value of the crossing energy levels is the same as the usual tortoise coordinate transformation.

2. METRIC OF ARBITRARILY ACCELERATING CHARGED BLACK HOLE

Using the advanced Eddington coordinate and adopting (−,+,+,+) signature, the line element of the arbitrarily accelerating charged black hole is:

$$ds^2 = g_{00}dv^2 + 2g_{01}dvdr + 2g_{02}dvd\theta + 2g_{03}dvd\varphi + g_{22}d\theta^2 + g_{33}d\varphi^2, \quad (1)$$

where

$$g_{00} = -\left[1 - \frac{2m}{r} - 2ar\cos\theta + \frac{Q^2}{r^2} - 4a\frac{Q^2}{r} - r^2(f^2 + h^2\sin^2\theta)\right],$$

$$g_{01} = g_{10} = 1, \quad g_{02} = g_{20} = r^2 f, \quad g_{03} = g_{30} = r^2 h\sin^2\theta,$$

$$g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta,$$

$$f = -a\sin\theta + b\sin\varphi + c\cos\varphi, h = \cot\theta(b\cos\varphi - c\sin\varphi), \quad (2)$$

$m = m(v)$, $Q = Q(v)$ are the black hole's mass and charge respectively. $a = a(v)$ is the magnitude of acceleration, while $b = b(v)$ and $c = c(v)$ describe the changing rate of the acceleration's direction. The determinant and non-zero contravariant components of the metric are:

$$g = -r^4\sin^2\theta, \quad (3)$$

$$g^{01} = g^{10} = 1, \quad g^{11} = 1 - \frac{2m}{r} - 2ar\cos\theta + \frac{Q^2}{r^2} - 4a\frac{Q^2}{r}\cos\theta,$$

$$g^{12} = g^{21} = -f, \quad g^{13} = g^{31} = -h, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2\sin^{2\theta}}. \quad (4)$$

The surface equation of event horizon can be written as:

$$H = H(v, r, \theta, \varphi) = 0, \quad (5)$$

or

$$r_h = r_h(v, \theta, \varphi), \quad (6)$$

which should satisfy null surface condition:

$$g^{\mu\nu}\frac{\partial H}{\partial x^{\mu}}\frac{\partial H}{\partial x^{\nu}}=0\cdot \tag{7}$$

From Equations (5) and (6), we have:

$$\frac{\partial H}{\partial r}\frac{\partial r}{\partial \theta}+\frac{\partial H}{\partial \theta}=0,\qquad \frac{\partial H}{\partial r}\frac{\partial r}{\partial \varphi}+\frac{\partial H}{\partial \varphi}=0,\qquad \frac{\partial H}{\partial r}\frac{\partial r}{\partial v}+\frac{\partial H}{\partial v}=0\cdot \tag{8}$$

From Equations (4), (7) and (8),we know that the event horizon should satisfy

$$1-\frac{2m}{r_h}-2ar_h\cos\theta+\frac{Q^2}{r_h^2}-4a\frac{Q^2}{r_h}\cos\theta-2r_{hv}+2fr_{h\theta}+2hr_{h\varphi}$$
$$+\frac{r_{h\theta}^2}{r_h^2}+\frac{r_{h\varphi}^2}{r_h^2\sin^2\theta}=0, \tag{9}$$

where

$$r_{h\theta}=\left(\frac{\partial r}{\partial \theta}\right)_{r=r_h},\qquad r_{hv}=\left(\frac{\partial r}{\partial v}\right)_{r=r_h},\qquad r_{h\varphi}=\left(\frac{\partial r}{\partial \varphi}\right)_{r=r_h}\cdot \tag{10}$$

Define a function as

$$F(r_h)=1-\frac{2m}{r_h}-2ar_h\cos\theta+\frac{Q^2}{r_h^2}-4a\frac{Q^2}{r_h}\cos\theta-2r_{hv}+2fr_{h\theta}+2hr_{h\varphi}$$
$$+\frac{r_{h\theta}^2}{r_h^2}+\frac{r_{h\varphi}^2}{r_h^2\sin^2\theta}, \tag{11}$$

we see $F(r_h)=0$ is identical with the event horizon Equation (9).

3. TEMPERATURE OF BLACK HOLE

According to Zhu *et al.* (1994), we can study the Hawking radiation of the arbitrarily accelerating charged black hole, and get the Hawking temperature. In the curved space-time, the dynamic equation of Klein-Gordon particle with mass μ is:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^{\mu}}-ieA_{\mu}\right)\left[\sqrt{-g}g^{\mu\nu}\left(\frac{\partial}{\partial x^{\nu}}-ieA_{\nu}\right)\Phi\right]-\mu^2\Phi=0, \tag{12}$$

where e is the charge of K-G particle, and A_{μ} is the electromagnetic four-vector produced by the black hole's charge.

Using the Lorentz condition $\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}g^{\mu\nu}A_{\nu}) = 0$,we have

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\Phi\right) - 2ieA_{\mu}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\Phi - e^{2}g^{\mu\nu}A_{\mu}A_{\nu}\Phi - \mu^{2}\Phi = 0\cdot \tag{13}$$

Equation (13) can be expressed by

$$g^{11}\frac{\partial^2\Phi}{\partial r^2} + 2\frac{\partial^2\Phi}{\partial v\partial r} + 2g^{12}\frac{\partial^2\Phi}{\partial r\partial\theta} + 2g^{13}\frac{\partial^2\Phi}{\partial r\partial\varphi} + g^{22}\frac{\partial^2\Phi}{\partial\theta^2} + g^{33}\frac{\partial^2\Phi}{\partial\varphi^2}$$
$$+ f_v\frac{\partial\Phi}{\partial v} + f_r\frac{\partial\Phi}{\partial r} + f_\theta\frac{\partial\Phi}{\partial\theta} + f_\varphi\frac{\partial\Phi}{\partial\varphi} + f_0\Phi = 0, \tag{14}$$

where

$$f_v = \frac{2}{r} - 2ieA_1,$$
$$f_r = \frac{2}{r}g^{11} + \frac{\partial g^{11}}{\partial r} + g^{12}\cot\theta + \frac{\partial g^{12}}{\partial\theta} + \frac{\partial g^{13}}{\partial\varphi}$$
$$-2ie(A_0 + A_1g^{11} + A_2g^{12} + A_3g^{13}),$$
$$f_\theta = \frac{2}{r}g^{12} + g^{22}\cot\theta - 2ie(A_1g^{12} + A_2g^{22}),$$
$$f_\varphi = \frac{2}{r}g^{13} - 2ie(A_1g^{13} + A_3g^{33}),$$
$$f_0 = -e^2g^{\mu\nu}A_\mu A_\nu - \mu^2\cdot \tag{15}$$

Now we use a new tortoise coordinate transformation

$$r_* = \frac{1}{2\kappa(v_0,\theta_0,\varphi_0)}\ln[r - r_h(v,\theta,\varphi)], \quad v_* = v - v_{0,}$$
$$\theta_* = \theta - \theta_0, \varphi_* = \varphi - \varphi_0, \tag{16}$$

then the Equation (14) become

$$\frac{\hat{g}^{11}}{2\kappa(r-r_h)}\frac{\partial^2\Phi}{\partial r_*^2} + 2\frac{\partial^2\Phi}{\partial v_*\partial r_*} + (2g^{12} - 2g^{22}r_{h\theta})\frac{\partial^2\Phi}{\partial\theta_*\partial r_*}$$
$$+(2g^{13} - 2g^{33}r_{h\varphi})\frac{\partial^2\Phi}{\partial\varphi_*\partial r_*} + \left[-\frac{\hat{g}^{11}}{r-r_h} - (g^{22}r_{h\theta\theta} + g^{33}r_{h\varphi\varphi})\right.$$
$$\left. - f_vr_{hv} + f_r - f_\theta r_{h\theta} - f_\varphi r_{h\varphi}\right]\frac{\partial\Phi}{\partial r_*} + 2\kappa(r-r_h)\left(g^{22}\frac{\partial^2}{\partial\theta_*^2} + g^{33}\frac{\partial^2}{\partial\varphi_*^2}\right.$$
$$\left. + f_v\frac{\partial}{\partial v_*} + f_\theta\frac{\partial}{\partial\theta_*} + f_\varphi\frac{\partial}{\partial\varphi_*} + f_0\right)\Phi = 0, \tag{17}$$

where

$$\hat{g}^{11} = g^{11} - 2r_{hv} - 2g^{12}r_{h\theta} - 2g^{13}r_{h\varphi} + g^{22}r_{h\theta}^2 + g^{33}r_{h\varphi}^2 \cdot \tag{18}$$

Comparing Equation (18) with Equation (11), we get

$$\lim_{r\to r_h} \hat{g}^{11} = F(r_h) = 0 \cdot \tag{19}$$

As $r \to r_h$ (it represents $r \to r_h(v_0, \theta_0, \varphi_0)$, $v \to v_0, \theta \to \theta_0, \varphi \to \varphi_0$), we suppose the coefficient of $\frac{\partial^2 \Phi}{\partial r_*^2}$ to be A, so

$$A = \lim_{r\to r_h} \frac{\hat{g}^{11}}{2\kappa\,(r - r_h)} = \lim_{r\to r_h} \frac{\partial \hat{g}^{11}/\partial r}{2\kappa} = \frac{1}{2\kappa} \lim_{r\to r_h} \frac{\partial \hat{g}^{11}}{\partial r} \cdot \tag{20}$$

Adjusting the parameter κ to make $A = 1$, so

$$\kappa = \frac{1}{2} \lim_{r\to r_h} \frac{\partial \hat{g}^{11}}{\partial r} = \frac{1}{r_h^2}(m + 2aQ^2\cos\theta) - \frac{1}{r_h^3}\left(Q^2 + r_{h\theta}^2 + \frac{r_{h\varphi}^2}{\sin^2\theta}\right) - a\cos\theta \cdot \tag{21}$$

As $r \to r_h$, Equation (18) can be reduced to

$$\frac{\partial^2 \Phi}{\partial r_*^2} + 2\frac{\partial^2 \Phi}{\partial v_* \partial r_*} + B\frac{\partial^2 \Phi}{\partial \theta_* \partial r_*} + C\frac{\partial^2 \Phi}{\partial \varphi_* \partial r_*} + (D + i2\omega_0)\frac{\partial \Phi}{\partial r_*} = 0, \tag{22}$$

where

$$B = \lim_{r\to r_h} (2g^{12} - 2g^{22}r_{h\theta}), \tag{23}$$

$$C = \lim_{r\to r_h} (2g^{13} - 2g^{33}r_{h\varphi}), \tag{24}$$

$$D = \lim_{r\to r_h} \left[-\frac{\hat{g}^{11}}{r - r_h} - \left(g^{22}r_{h\theta\theta} + g^{33}r_{h\varphi\varphi} + \frac{2r_{hv}}{r} \right) + \left(\frac{2}{r}g^{11} + \frac{\partial g^{11}}{\partial r} + \frac{\partial g^{12}}{\partial \theta} + \frac{\partial g^{13}}{\partial \varphi} \right) - \left(\frac{2}{r}g^{12} + g^{22}\cot\theta \right) r_{h\theta} - \frac{2}{r}g^{13}r_{h\varphi} \right], \tag{25}$$

$$\omega_0 = \lim_{r\to r_h} [eA_1 r_{hv} + e(A_1 g^{12} + A_2 g^{22})r_{h\theta} + e(A_1 g^{13} + A_3 g^{33})r_{h\varphi} - e(A_0 + A_1 g^{11} + A_2 g^{12} + A_3 g^{13})] \cdot \tag{26}$$

Obviously $B,\ C,\ D,\ \omega_0$ have limited values respectively.

In the vicinity of the event horizon, the solution of Equation (22) can be written as (Zhao and Dai, 1991)

$$\Phi = R\,(r_*)\, e^{-i\omega v_* + ik_\theta \theta_* + ik_\varphi \varphi_*}. \tag{27}$$

Substituting Equation (27) into Equation (22), we obtain

$$\frac{\partial^2 R(r_*)}{\partial r_*^2} + \left\{D - i\left[2(\omega-\omega_0) - Bk_\theta - Ck_\varphi\right]\right\}\frac{\partial R(r_*)}{\partial r_*} = 0\cdot \quad (28)$$

The solutions of Equation (28) are

$$R_\omega^{\text{in}} = e^{-i\omega v_*}, \qquad R_\omega^{\text{out}} = e^{-i\omega v_*}e^{-Dr_* + i\left[2(\omega-\omega_0)-Bk_\theta - Ck_\varphi\right]r_*}. \quad (29)$$

So the in-going wave and the out-going wave on horizon are

$$\begin{aligned}\Phi_\omega^{\text{in}} &= e^{-i\omega v_*}e^{ik_\theta\theta_* + ik_\varphi\varphi_*},\\ \Phi_\omega^{\text{out}} &= e^{-i\omega v_*}e^{-Dr_* + i\left[2(\omega-\omega_0)-Bk_\theta - Ck_\varphi\right]r_*}e^{ik_\theta\theta_* + ik_\varphi\varphi_*}\\ &= e^{-i\omega v_*}(r-r_h)^{-D/2\kappa}(r-r_h)^{i\left[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi\right]/2k}e^{ik_\theta\theta_* + ik_\varphi\varphi_*}.\end{aligned} \quad (30)$$

We see that Φ_ω^{out} is not analytic at $r \to r_h$, so we have to analytically extend it through the lower half complex r plane into the inside of the event horizon and obtain

$$r - r_h = |r - r_h|\, e^{-i\pi} = (r_h - r)\, e^{-i\pi}\cdot$$

So the out-going wave at $r < r_h$ is

$$\begin{aligned}\Phi_\omega^{\text{out}}(r < r_h) &= e^{-i\omega v_* + ik_\theta\theta_* + ik_\varphi\varphi_*}\left[(r_h - r)e^{-i\pi}\right]^{-D/2\kappa}\\ &\quad\times\left[(r_h - r)e^{-i\pi}\right]^{i[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/2\kappa}\\ &= e^{-i\omega v_*}e^{-Dr_* + i[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]r_*}e^{ik_\theta\theta_* + ik_\varphi\varphi_* + iD\pi/2\kappa}\\ &\quad\times e^{\pi[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi]/2\kappa}.\end{aligned} \quad (31)$$

The probability of the scattered out-going wave is

$$\left|\frac{\Phi_\omega^{\text{out}}(r > r_h)}{\Phi_\omega^{\text{out}}(r < r_h)}\right|^2 = e^{-\pi\left[2(\omega-\omega_0)-Bk_\theta-Ck_\varphi\right]/\kappa}. \quad (32)$$

According to Damour and Ruffini (1976) and Sannan (1998), we get the Hawking thermal spectrum as

$$N_\omega = \frac{1}{e^{\left[(\omega-\omega_0)-Bk_\theta/2-Ck_\varphi/2\right]/k_B T} - 1}, \quad (33)$$

where k_{B} is Boltzmann's constant. This result indicates that the parameter κ is connected with the radiation temperature by

$$T = \frac{\kappa}{2\pi k_{\text{B}}}\cdot \quad (34)$$

4. THE DRAGGING VELOCITIES

Introducing a coordinate transformation

$$R = r - r_h, \qquad dR = dr - r_{hv}dv - r_{h\theta}d\theta - r_{h\varphi}d\varphi, \tag{35}$$

the line element (1) becomes

$$ds^2 = \hat{g}_{00}dv^2 + 2\,dvdR + 2\hat{g}_{02}dvd\theta + 2\hat{g}_{03}dvd\varphi + \hat{g}_{22}d\theta^2 + \hat{g}_{33}d\varphi^2, \tag{36}$$

where

$$\hat{g}_{00} = g_{00} + 2r_{hv}, \qquad \hat{g}_{02} = g_{02} + r_{h\theta},$$
$$\hat{g}_{03} = g_{03} + r_{h\varphi}, \qquad \hat{g}_{22} = g_{22}, \qquad \hat{g}_{33} = g_{33}\cdot \tag{37}$$

From Equation (36), we have

$$ds^2 = \left(\hat{g}_{00} - \frac{\hat{g}_{02}^2}{\hat{g}_{22}} - \frac{\hat{g}_{03}^2}{\hat{g}_{33}}\right)dv^2 + 2dvdR + \hat{g}_{22}\left(d\theta + \frac{\hat{g}_{02}}{\hat{g}_{22}}dv\right)^2$$
$$+\,\hat{g}_{33}\left(d\varphi + \frac{\hat{g}_{03}}{\hat{g}_{33}}dv\right)^2\cdot \tag{38}$$

Obviously, the dragging velocities are

$$\Omega_\theta = \frac{d\theta}{dv} = -\frac{\hat{g}_{02}}{\hat{g}_{22}}, \qquad \Omega_\varphi = \frac{d\varphi}{dv} = -\frac{\hat{g}_{03}}{\hat{g}_{33}}\cdot \tag{39}$$

Comparing Equations (23) , (24) with Equation (38), we obtain

$$\frac{B}{2} = (\Omega_\theta)_{r\to r_h}, \qquad \frac{C}{2} = \left(\Omega_\varphi\right)_{r\to r_h}\cdot \tag{40}$$

We see that the parameters B and C in Equations (23) and (24) are concerned with the dragging velocities on the event horizon.

5. ENTROPY OF THE BLACK HOLE

According to the method suggested by Tian and Zhao (2002), we use the thin film model to calculate the entropy of black hole. In the thin film model , the black hole entropy are contribution of the field near the horizon in region $r_h + \varepsilon \to r_h + \varepsilon + \delta$($\varepsilon$ is the distance from the horizon, δ is the thickness). The temperature is variable with the position or the angles (θ, φ), therefore we must decompose the region into many small parts: for each part, the region is $r_h\,(v, \theta_i, \varphi_i) + \varepsilon \to r_h\,(v, \theta_i, \varphi_i) + \varepsilon + \delta, \theta_i = \theta_i + \triangle\theta_i, \varphi_i = \varphi_i + \triangle\varphi_i.\, i = 1, 2, 3, \ldots, n, \ldots,$ and we consider that the field is quasi-equilibrium and the statistical mechanics is valid in each small region.

The determinant and the non-zero contra-variant components of the metric (36) are

$$\hat{g} = -r^4 \sin^2\theta, \tag{41}$$

$$\begin{aligned}
&\hat{g}^{01} = 1,\, \hat{g}^{22} = \frac{1}{r^2},\, \hat{g}^{33} = \frac{1}{r^2\sin^2\theta},\\
&\hat{g}^{12} = -\frac{\hat{g}_{02}}{\hat{g}_{22}} = \Omega_\theta,\, \hat{g}^{13} = -\frac{\hat{g}_{03}}{\hat{g}_{33}} = \Omega_\varphi,\\
&\hat{g}^{11} = 1 - \frac{2m}{r} - 2ar_h\cos\theta + \frac{Q^2}{r^2} - 4a\frac{Q^2}{r}\cos\theta - 2r_{hv} + 2fr_{h\theta} + 2hr_{h\varphi}\\
&\quad + \frac{r_{h\theta}^2}{r^2} + \frac{r_{h\varphi}^2}{r^2\sin^2\theta}\cdot
\end{aligned} \tag{42}$$

We see that $\hat{g}^{11}$ here is just that in Equation (18) , so we have

$$\hat{g}^{11}(r_h) = 0,\, \frac{\partial \hat{g}^{11}}{\partial r}|_{r=r_h} = 2\kappa\cdot \tag{43}$$

The K-G equation of the metric in Equation (36) is

$$\frac{1}{\sqrt{-\hat{g}}}\left(\frac{\partial}{\partial x^\mu} - ie\hat{A}_\mu\right)\left[\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\left(\frac{\partial}{\partial x^\nu} - ie\hat{A}_\nu\right)\right]\Phi - \mu^2\Phi = 0, \tag{44}$$

where $\hat{A}_\mu$ is the transformation of A_μ in the original metric. Supposing $\Phi = e^{-iEv+iG(R,\theta,\varphi)}$ with WKB approximation, using Lorentz condition,we get

$$\begin{aligned}
&\hat{g}^{11}k_R^2 - 2[E - \Omega_\theta k_\theta - \Omega_\varphi k_\varphi + e(\hat{A}_0 + \hat{A}_1\hat{g}^{11} + \hat{A}_2\Omega_\theta + \hat{A}_3\Omega_\varphi)]k_R + \hat{g}_{22}k_\theta^2\\
&\quad + \hat{g}^{33}k_\varphi^2 - 2e(\hat{A}_1\Omega_\theta + \hat{A}_2\hat{g}^{22})k_\theta - 2e(\hat{A}_1\Omega_\varphi + \hat{A}_3\hat{g}^{33})k_\varphi + 2e\hat{A}_1E\\
&\quad + e^2(2\hat{A}_0\hat{A}_1 + 2\Omega_\theta\hat{A}_1\hat{A}_2 + 2\Omega_\varphi\hat{A}_1\hat{A}_3 + \hat{g}^{11}\hat{A}_1^2 + \hat{g}^{22}\hat{A}_2^2 + \hat{g}^{33}\hat{A}_3^2) + \mu^2 = 0,
\end{aligned} \tag{45}$$

where $k_R = \frac{\partial G}{\partial R},\ k_\theta = \frac{\partial G}{\partial \theta},\ k_\varphi = \frac{\partial G}{\partial \varphi}\cdot$

The solutions of Equation (45) are

$$\begin{aligned}
k_R^\pm = &\frac{E' + e\hat{A}_1\hat{g}^{11}}{\hat{g}^{11}}\\
&\pm\frac{1}{\hat{g}^{11}}\sqrt{E^2 - \hat{g}^{11}[\hat{g}^{22}(k_\theta - e\hat{A}_2)^2 + \hat{g}^{33}(k_\varphi - e\hat{A}_3)^3 + \mu^2]},
\end{aligned} \tag{46}$$

where $E' = E - \Omega_\theta k_\theta - \Omega_\varphi k_\varphi + e\left(\hat{A}_0 + \hat{A}_2\Omega_\theta + \hat{A}_3\Omega_\varphi\right)\cdot$

According to quantum statistical mechanics, decomposing the thin film into many small parts, the free energy of No. i subsystem is given by

$$\triangle F_i = -\int_0^\infty dE' \frac{\Gamma(E')}{e^{\beta E'} - 1}, \tag{47}$$

where $\Gamma\left(E'\right)$ is the number of quantum states with energy less than E'. According to the quasi-classical quantized condition and thin film brick-wall model, we have

$$\Gamma\left(E'\right) = \frac{1}{4\pi^3}\int dk_\theta \int dk_\varphi \int_{\theta i}^{\theta i+\Delta\theta i} d\theta \int_{\varphi_i}^{\varphi_i+\Delta\varphi_i} d\varphi \left(\int_{\varepsilon}^{\varepsilon+\delta} k_R^+ dR + \int_{\varepsilon+\delta}^{\varepsilon} k_R^- dR\right). \tag{48}$$

Substituting Equation (46) into Equation (48), and considering that $E'^2 - \hat{g}^{11}[\hat{g}^{22}(k_\theta - e\hat{A}_2)^2 + \hat{g}^{33}(k_\varphi - e\hat{A}_3)^2 + \mu^2] \geq 0$ restricts the upper limits and the lower limits of k_θ, k_φ, so

$$\begin{aligned}\Gamma\left(E'\right) &= \frac{E'^3}{6\pi^2}\int_{\theta_i}^{\theta_i+\Delta\theta i} d\theta \int_{\varphi_i}^{\varphi i+\Delta\varphi i} d\varphi \int_{\varepsilon}^{\varepsilon+\delta} \left(\hat{g}^{11}\right)^{-2} (\hat{g}^{22}\hat{g}^{33})^{-\frac{1}{2}} dR \\ &= \frac{E'^3}{6\pi^2}\int dA_i \int_{\varepsilon}^{\varepsilon+\delta} (\hat{g}^{11})^{-2} dR,\end{aligned} \tag{49}$$

where $\int dA_i = \int_{\theta i}^{\theta i+\Delta\theta i}\int_{\varphi i}^{\varphi_i+\Delta\varphi_i} (\hat{g}^{22}\hat{g}^{33})^{-1/2} d\theta d\varphi$ is the small area on the event horizon in the No. i subsystem. We rewrite it as $\triangle A_i$.

Substituting Equation (49) into Equation (48),we can obtain

$$\triangle F_i = -\frac{\triangle A_i}{6\pi^2}\int_{\varepsilon}^{\varepsilon+\delta} (\hat{g}^{11})^{-2} dR \int_0^\infty \frac{E'^3}{e^{\beta E'} - 1} dE' \cdot \tag{50}$$

As $\hat{g}^{11}\left(r_h\right) = 0$, we can decompose $\hat{g}^{11}$ as $\hat{g}^{11} = p\left(v, r, \theta, \varphi\right)\left(r - r_h\right)$, so

$$\begin{aligned}\triangle F_i &= -\frac{\triangle A_i}{6\pi^2}\int_0^\infty \frac{E'^3}{e^{\beta E'} - 1} dE' \int_{\varepsilon}^{\varepsilon+\delta} \frac{1}{p^2\left(v, r, \theta, \varphi\right)\left(r - r_h\right)^2} dR \\ &\approx -\triangle A_i \frac{\pi^2\delta}{90\beta^4 p^2\left(r_h\right)\varepsilon\left(\varepsilon+\delta\right)}.\end{aligned} \tag{51}$$

From the relation of the entropy and the free energy of the subsystem

$$\triangle S_i = \beta^2 \frac{\partial \triangle F_i}{\partial \beta}|_{\beta=\beta_h} = -\triangle A_i \frac{4\pi^2}{90\beta_h^4 p^2\left(r_h\right)} \frac{\delta}{\varepsilon\left(\varepsilon+\delta\right)}. \tag{52}$$

Considering Equation (43), we have $p\left(r_h\right) = \frac{\partial \hat{g}^{11}}{\partial r}|_{r=r_h} = 2\kappa$, noticing $\beta_h = \frac{2\pi}{\kappa}$, then

$$\triangle S_i = \frac{1}{90\beta_h} \frac{\delta}{\varepsilon\left(\varepsilon+\delta\right)} \frac{1}{4} \triangle A_i\cdot \tag{53}$$

If we choose ε, δ to make

$$\frac{\delta}{\varepsilon\left(\varepsilon+\delta\right)} = 90\beta_h, \tag{54}$$

we can get

$$\triangle S_i = \frac{1}{4} \triangle A_i\cdot \tag{55}$$

So the total entropy of the black hole is

$$S = \sum_i \triangle S_i = \frac{1}{4}\sum_i \triangle A_i = \frac{1}{4} A_h\cdot \tag{56}$$

6. NONTHERMAL RADIATION OF THE BLACK HOLE

In the curved space-time described by the line element (1), the Hamilton–Jacobin equation of the particle with mass μ and charge e is (Yang and Zhao, 1993)

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - eA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - eA_\nu\right) + \mu^2 = 0\cdot \tag{57}$$

Considering the new tortoise coordinate transformation such as (16),we have

$$\tilde{A}\left(\frac{\partial S}{\partial r_*}\right)^2 + 4\kappa\left(r-r_h\right)\tilde{B}\left(\frac{\partial S}{\partial r_*}\right) + \left[2\kappa\left(r-r_h\right)\right]^2 \tilde{C} = 0, \tag{58}$$

where

$$\tilde{A} = -2r_{hv} + g^{11} - 2g^{12} r_{h\theta} - 2g^{13} r_{h\varphi} + g^{22} r_{h\theta}^2 + g^{33} r_{h\varphi}^2, \tag{59}$$

$$\begin{aligned}\tilde{B} = & -\omega - eA_0 + \left(r_{hv} + g^{12} r_{h\theta} + g^{13} r_{h\varphi} - g^{11}\right) eA_1 + \left(g^{22} r_{h\theta} - g^{12}\right) eA_2 \\ & + (g^{33} r_{h\varphi} - g^{13}) eA_3 + (g^{12} - g^{22} r_{h\theta}) P_\theta + (g^{13} - g^{33} r_{h\varphi}) P_\varphi,\end{aligned} \tag{60}$$

$$\begin{aligned}\tilde{C} = & \, 2eA_1\omega + e^2\left(2A_0A_1 + g^{11}A_1^2 + 2g^{12}A_1A_2 + 2g^{13}A_1A_3 + g^{22}A_2^2 + g^{33}A_3^2\right) \\ & - 2e(g^{12}A_1 + g^{22}A_2)P_\theta - 2e(g^{13}A_1 + g^{33}A_3)P_\varphi + g^{22}P_\theta^2 + g^{33}P_\varphi^2 + \mu^2,\end{aligned} \tag{61}$$

$$\omega = -\frac{\partial S}{\partial v_*}, \quad P_\theta = \frac{\partial S}{\partial \theta_*}, \quad P_\varphi = \frac{\partial S}{\partial \varphi_*}. \tag{62}$$

The solution of Equation (58) is

$$\frac{\partial S}{\partial r_*} = \frac{-2\kappa\,(r - r_h)\,\tilde{B} \pm 2\kappa\,(r - r_h)\,(\tilde{B}^2 - \tilde{A}\tilde{C})^{\frac{1}{2}}}{\tilde{A}}, \tag{63}$$

where S is the Hamilton principal function, and $\frac{\partial S}{\partial x^\mu}$ is the generalized four-momentum, so both S and $\frac{\partial S}{\partial r_*}$ are real numbers. We can obtain

$$\left(\tilde{B}^2 - \tilde{A}\tilde{C}\right) \geq 0\cdot \tag{64}$$

Because $\tilde{B}$ and$\tilde{C}$ in Equations (60) and (61) are concerned with energy ω , Equation (64) is the equation that the particle's energy level should be satisfied.

Supposing $\tilde{D} = \tilde{B} + \omega$, $\tilde{E} = \tilde{C} - 2eA_1\omega$, Equation (64) becomes

$$(\tilde{D} - \omega)^2 - \tilde{A}(\tilde{E} + 2eA_1\omega) \geq 0\cdot \tag{65}$$

We select qual sign in Equation (65) ,the solutions are

$$\omega^{\pm} = (\tilde{D} + e\tilde{A}A_1) \pm (2e\tilde{A}A_1\tilde{D} + e^2\tilde{A}^2A_1^2 + \tilde{A}\tilde{E})^{\frac{1}{2}}\cdot \tag{66}$$

This is the Dirac energy level distribution of the scalar particle. $\omega \geq \omega^+$ is the positive energy state, and $\omega \leq \omega^-$ is the negative energy state. The state of the particle with energy $\omega^- < \omega < \omega^+$ is the forbidden area, and the width of the forbidden area is

$$\triangle\omega = \omega^+ - \omega^- = 2(2e\tilde{A}A_1\tilde{D} + e^2\tilde{A}^2A_1^2 + \tilde{A}\tilde{E})^{\frac{1}{2}}\cdot \tag{67}$$

Comparing Equation (59) with Equations (18) and (19), we have $\tilde{A}|_{r\to r_h} = \hat{g}^{11}|_{r\to r_h} = 0$, so

$$\triangle\omega|_{r\to r_h} = 0, \tag{68}$$

$$\begin{aligned}\tilde{\omega_0} &= \omega^+|_{r\to r_h} = \omega^-|_{r\to r_h} = \tilde{D}|_{r\to r_h} \\ &= [eA_1 r_{hv} + e(A_1g^{12} + A_2g^{22})r_{h\theta} + e(A_1g^{13} + A_3g^{33})r_{h\varphi} - e(A_0 \\ &\quad + A_1g^{11} + A_2g^{12} + A_3g^{13})]|_{r\to r_h} + (g^{12} - g^{22}r_{h\theta})|_{r\to r_h}(P_\theta)_{r\to r_h} \\ &\quad + (g^{13} - g^{33}r_{h\varphi})|_{r\to r_h}(P_\varphi)_{r\to r_h}\cdot\end{aligned} \tag{69}$$

Considering Equations (23) , (24) and (26),we get

$$\tilde{\omega_0} = \omega_0 + \frac{B\,(P_\theta)_{r\to r_h}}{2} + \frac{C\,\left(P_\varphi\right)_{r\to r_h}}{2}\cdot \tag{70}$$

This equation gives us the maximum value of the crossing energy level of a particle, which is the same as the usual tortoise coordinate transformation.

The Hawking thermal spectrum in Equation (33) is

$$N_\omega = \frac{1}{e^{\left[(\omega-\omega_0)-Bk_\theta/2-Ck_\varphi/2\right]/k_B T} - 1}$$
$$= -\frac{1}{e^{\left[\omega-\left(\omega_0+Bk_\theta/2+Ck_\varphi/2\right)\right]/k_B T} - 1}.$$

Using Equation (41), we obtain

$$\omega_0 + \frac{Bk_\theta}{2} + \frac{Ck_\varphi}{2} = \omega_0 + k_\theta(\Omega_\theta)_{r\to r_h} + k_\varphi(\Omega_\varphi)_{r\to r_h}. \tag{71}$$

This is the chemical potential. According to the de Brogile relation in the quantum mechanics

$$(P_\theta)_{r\to r_h} = \hbar k_\theta, \qquad (P_\varphi)_{r\to r_h} = \hbar k_\varphi, \tag{72}$$

in the natural unite system $h = 1$, so the chemical potential in Equation (71) accords with that in Equation (70).

The phenomenon of the crossing of the particle with positive or negative energy happens near the event horizon, and the maximum value of the crossing energy levels is $\tilde{\omega}_0$. Only if the energy of the particle in the negative energy state, is higher than the lowest energy of the particle in the positive energy state, it can radiate out through the forbidden area by tunnel effect. This is the non-thermal radiation of the black hole.

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REFERENCES

Bekenstein, J. D. (1973). *Physical Review D* **7**, 2333.
Damour, T. and Ruffini, R. (1976). *Physical Review D* **14**, 332.
Hawking, S. W. (1975). *Communications in Mathematical Physics* **43**, 199.
He, H., Zhao, Z., and Zhang, L. H. (2002). *International Journal of Theoretical Physics* **41**, 1781.
Li, Z. H. and Zhao, Z. (1997) *Acta Physica Sinica* **46**, 1273 (in Chinese).
Li, X. and Zhao, Z. (2001). *Chinese Physics Letters* **18**, 463.
Sannan, S. (1998). *General Relativity and Gravitation* **20**, 239.
‘t Hooft, G. (1985). *Nuclear Physics B* **256**, 727.
Tian, Guihua and Zhao, Zheng (2002). *Nuclear Physics B* **636**, 418.

Yang, S. Z. and Zhao, Z. (1993). *Journal of Beijing Normal University (Natural Science)* **29** (supplement 1), 42.

Yang, Xuejun (2003). *Study on black hole thermodynamics and Killing reduction of spacetime*, doctor dissertation, Department of Physics. Beijing Normal University, Beijing.

Zhao, Z. and Dai, X. X. (1991). *Chinese Physics Letters* **8**, 548.

Zhu, J. Y., Zhang, J. H., and Zhao, Z. (1994). *Acta Astronomica Sinica* **35**, 426 (in Chinese).